

Kai Cieliebak¹ and Edward Goldstein²

A NOTE ON MEAN CURVATURE, MASLOV CLASS AND SYMPLECTIC AREA OF LAGRANGIAN IMMERSIONS

ABSTRACT. In this note we prove a simple relation between the mean curvature form, symplectic area, and the Maslov class of a Lagrangian immersion in a Kähler-Einstein manifold. An immediate consequence is that in Kähler-Einstein manifolds with positive scalar curvature, minimal Lagrangian immersions are monotone.

1. INTRODUCTION

Let (M, ω) be a Kähler-Einstein manifold whose Ricci curvature is a multiple of the metric by a real number λ . Hence the Kähler form ω and the first Chern class $c_1(M)$ are related by $c_1(M) = \lambda[\omega]$. Let L be an immersed Lagrangian submanifold of M and let σ_L be the mean curvature form of L (which is a closed 1-form on L). Let $F : \Sigma \rightarrow M$ be a smooth map from a compact connected surface to M whose boundary ∂F is contained in L . Let $\mu(F)$ be the Maslov class of F and $\omega(F)$ its symplectic area. The goal of this note is to prove the following simple relation between these quantities:

$$(1) \quad \mu(F) - 2\lambda\omega(F) = \frac{\sigma_L(\partial F)}{\pi}.$$

This relation was given in [M] for \mathbb{C}^n and in [Ar] for Calabi-Yau manifolds. Dazord [D] showed that the differential of the mean curvature form is the Ricci form, so in the Kähler-Einstein case σ_L is closed. Y.G. Oh [Oh2] investigated the symplectic area in the case that the mean curvature form is exact.

In the case $\lambda > 0$, Lagrangian submanifolds for which the left-hand side vanishes on all disks F are called *monotone* in the symplectic geometry literature, cf. [Oh1]. An immediate consequence of (1) is that in Kähler-Einstein manifolds with positive scalar curvature, minimal Lagrangian immersions are monotone.

In view of the condition $c_1(M) = \lambda[\omega]$, the left-hand side of (1) depends only on the boundary of F . Thus if the map $H_1(L; \mathbb{R}) \rightarrow H_1(M; \mathbb{R})$ is trivial it defines a cohomology class $\delta_L \in H^1(L; \mathbb{R})$ via $\delta_L(\gamma) := \mu(F) - 2\lambda\omega(F)$ for some 2-cycle F with $\partial F = \gamma$. It follows that in this case the cohomology class of the mean curvature form σ_L is invariant under symplectomorphisms of M . This generalizes Oh's observation [Oh2] that the cohomology class is invariant under Hamiltonian deformations. One consequence is the following:

Let (M, ω) be a Kähler manifold with $c_1(M) = \lambda[\omega] \in H_2(M; \mathbb{R})$. Let L be an immersed Lagrangian submanifold of M such that the map $H_1(L; \mathbb{R}) \rightarrow H_1(M; \mathbb{R})$ is trivial and $\delta_L \neq 0$. Suppose there is a Kähler-Einstein metric ω_{KE} in the same cohomology class as ω and $\phi : (M, \omega) \rightarrow (M, \omega_{KE})$ is a symplectomorphism (e.g. the one provided by Moser's theorem). Then $\phi(L)$ is Lagrangian but *not* minimal.

Note that for $\lambda \neq 0$ most Lagrangian submanifolds L with nontrivial first Betti number such that $H_1(L) \rightarrow H_1(M)$ vanishes have $\delta_L \neq 0$: For any such L , pick a

¹kai@mathematik.uni-muenchen.de

²egold@math.stanford.edu

normal vector field v to L such that $i_v\omega$ is closed on L and non-trivial cohomologically. Then small time variations of L through v produce Lagrangian submanifolds with nontrivial δ_L .

2. NOTATION

We first recall the definition of the Maslov index that is suitable for our purposes. Let V be a Hermitian vector space of complex dimension n . Let $\Lambda^{(n,0)}V$ be the (one-dimensional) space of holomorphic $(n, 0)$ -forms on V and set

$$K^2(V) := \Lambda^{(n,0)}V \otimes \Lambda^{(n,0)}V.$$

Let L be a Lagrangian subspace of V . We can associate to L an element $\kappa(L)$ in $\Lambda^{(n,0)}V$ of unit length which restricts to a real volume form on L . This element is unique up to sign and therefore defines a unique element of unit length

$$\kappa^2(L) := \kappa(L) \otimes \kappa(L) \in K^2(V).$$

Thus we get a map κ^2 from the Grassmanian $Gr_{\text{Lag}}(V)$ of Lagrangian planes to the unit circle in $K^2(V)$. This map induces a homomorphism κ_*^2 of fundamental groups

$$\kappa_*^2 : \pi_1(Gr_{\text{Lag}}(V)) \rightarrow \mathbb{Z}.$$

To understand the map κ_*^2 , let L be a Lagrangian subspace and let v_1, \dots, v_n be an orthonormal basis for L . For $0 \leq t \leq 1$ consider the subspace

$$L_t = \text{span}\{v_1, \dots, v_{n-1}, e^{\pi it}v_n\}.$$

This loop $\{L_t\}$ is the standard generator of $\pi_1(Gr_{\text{Lag}}(V))$. The induced elements in $\Lambda^{(n,0)}V$ are related by $\kappa(L_t) = \pm e^{-\pi it}\kappa(L)$, so $\kappa^2(L_t) = e^{-2\pi it}\kappa^2(L)$ and $\kappa_*^2(\{L_t\}) = -1$. Thus we see that the homomorphism κ_*^2 is related to the Maslov index μ (as defined, e.g., in [ALP]) by

$$\kappa_*^2 = -\mu : \pi_1(Gr_{\text{Lag}}(V)) \rightarrow \mathbb{Z}.$$

Now let (M, ω) be a symplectic manifold of dimension $2n$. Pick a compatible almost complex structure J on M and let $K(M)$ be the canonical bundle of M , i.e., $K(M) := \Lambda^{(n,0)}T^*M$ is the bundle of $(n, 0)$ -forms on M . Note that $c_1(K(M)) = -c_1(M)$. Let $K^2(M) := K(M) \otimes K(M)$ be the square of the canonical bundle.

Let L be an immersed Lagrangian submanifold of M . For any point $l \in L$ there is an element of unit length $\kappa(l)$ of $K(M)$ over l , unique up to sign, which restricts to a real volume form on the tangent space T_lL . The squares of these elements give rise to a section of unit length

$$\kappa_L^2 : L \rightarrow K^2(M).$$

Now let $F : \Sigma \rightarrow M$ be a smooth map with boundary ∂F on L . The *symplectic area* of F is

$$\omega(F) = \int_{\Sigma} F^*\omega.$$

This defines a map from the relative second homology group to \mathbb{R} ,

$$[\omega] : H_2(M, L; \mathbb{Z}) \rightarrow \mathbb{R}.$$

To define the Maslov class $\mu(F)$, choose a unitary frame for the tangent bundle TM along F . Consider the dual frame and wedge all its elements. Thus we get a

unit length section κ_F of $K(M)$ over F . Now on the boundary $\partial F = F(\partial\Sigma)$ we also have the section κ_L^2 defined above. We can uniquely write

$$\kappa_L^2 = e^{i\theta} \kappa_F^2$$

for a function $e^{i\theta} : \partial\Sigma \rightarrow S^1$ to the unit circle. The Maslov class $\mu(F)$ is minus its winding number,

$$\mu(F) := \frac{-1}{2\pi} \int_{\partial F} d\theta.$$

This defines a map

$$\mu : H_2(M, L; \mathbb{Z}) \rightarrow \mathbb{Z}.$$

In view of the discussion above, this definition agrees with the usual definition of the Maslov class, cf. [ALP].

Now suppose that $c_1(M) = \lambda[\omega] \in H_2(M; \mathbb{R})$. It is well-known that if ∂F is trivial in $H_1(L; \mathbb{R})$, then F represents an element in $[F] \in H_2(M; \mathbb{R})$ and

$$\mu(F) = 2c_1(M)([F]) = 2\lambda\omega(F).$$

So in this case $\mu(F) - 2\lambda\omega(F)$ depends only on the boundary $\partial F \in H_1(L; \mathbb{R})$. If, moreover, the map $H_1(L; \mathbb{R}) \rightarrow H_1(M; \mathbb{R})$ is trivial, this expression defines a cohomology class $\delta_L \in H^1(L; \mathbb{R})$ via

$$\delta_L(\gamma) := \mu(F) - 2\lambda\omega(F)$$

for some 2-cycle F with $\partial F = \gamma$.

3. PROOF

Now assume that (M, ω) is Kähler-Einstein, i.e., M carries a Kähler metric whose Ricci curvature is a multiple of the metric by a constant $\lambda \in \mathbb{R}$. This is equivalent to saying that the curvature form of the canonical bundle $K(M)$ equals $-\frac{2\pi}{i}\lambda\omega$. We denote the connections on $K(M)$ and $K^2(M)$ (induced by the Levi-Civita connection) by ∇ .

Let L be an immersed Lagrangian submanifold of M and let κ_L^2 be the canonical section of $K^2(M)$ over L as above. The section κ_L^2 defines a connection 1-form ξ_L for $K^2(M)$ over L by the condition $\nabla\kappa_L^2 = \xi_L \otimes \kappa_L^2$. Since κ_L^2 has constant length 1, ξ_L is an imaginary valued 1-form on L . From the Einstein condition and the fact that L is Lagrangian we get $d(i\xi_L) = -4\pi\lambda\omega|_L = 0$, so the form $i\xi_L$ is closed.

Let H be the trace of the second fundamental form of L (the mean curvature vector field of L). Thus H is a section of the normal bundle to L in M and we have a corresponding 1-form $\sigma_L := i_H\omega$ on L . The following fact goes back to [Oh2] (see also [Gold1] for a proof):

$$\sigma_L = i\xi_L/2.$$

(Here the factor 1/2 is due to the fact that ξ_L is a connection 1-form for $K^2(M)$ rather than $K(M)$.) Thus σ_L is a closed 1-form on L , called the *mean curvature form* on L .

Having explained all the terms in formula (1), we now turn to its proof. Let $F : \Sigma \rightarrow M$ be a smooth map from a compact surface with boundary on L . Define the section κ_F of $K(M)$ over F as above, using a unitary trivialization of TM over

F . Let ξ_F be the connection 1-form along F defined by $\nabla \kappa_F^2 = \xi_F \otimes \kappa_F^2$. The Einstein condition tells us that $d(i\xi_F) = -4\pi\lambda F^*\omega$. Thus by Stokes' theorem,

$$2\lambda\omega(F) = \int_{\partial F} \frac{-i\xi_F}{2\pi}.$$

Recall that along ∂F we have $\kappa_L^2 = e^{i\theta}\kappa_F^2$ for a function $e^{i\theta} : \partial\Sigma \rightarrow S^1$, and the Maslov class is given by

$$\mu(F) = -\frac{1}{2\pi} \int_{\partial F} d\theta.$$

The connection 1-forms ξ_F and ξ_L are related by

$$\xi_L = \xi_F + i d\theta.$$

Thus

$$\frac{\sigma_L(\partial F)}{\pi} = \int_{\partial F} \frac{i\xi_L}{2\pi} = \int_{\partial F} \frac{i\xi_F}{2\pi} - \int_{\partial F} \frac{d\theta}{2\pi} = \mu(F) - 2\lambda\omega(F).$$

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